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Damage integration in the strain space ☆

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Abstract

The focus is on isotropic elastodamaging (softening) materials, where the damage parameter is expressed as a function of the total strain. By integrating the mechanical work in the strain space along a stepwise holonomic loading history, an incremental strain energy is obtained. A coaxiality condition for the incremental strain energy to be potential is identified, and its implications on the associativity of the damage evolution are discussed. Under some hypotheses, the increment of the mechanical work is shown to be minimum along strain radial paths. These results are used to construct a multifield variational framework supporting finite element (nonlocal) formulations.

Keywords: Isotropic damage; Softening; Strain energy; Variational calculus; Nonconvex; Nonlocality

1. Introduction

Quasi-brittle materials such as concrete and geomaterials are often defined as elastodamaging. Structural elements made of elastodamaging materials exhibit a load–displacement response, where an elastic branch is followed by a peak in correspondence of a critical displacement and, eventually, by a softening branch along which the load decreases for increasing displacement. Such a structural behavior can be conveniently modelled by means of softening stress–strain laws and assuming the damage as a strain driven phenomenon. The material response is strongly path-dependent and globally nonholonomic. Hence, the entire evolution of the structural response is usually analyzed as a sequence of incremental problems, each concerning a configuration change from an initial known state due to a finite load step. The nonholonomic response can then be transformed through an implicit backward Euler integration scheme into its stepwise holonomic counterpart (Comi et al., 1992; Simo and Hughes, 1997; Tin-Loi and Xia, 2001). In the generalized standard materials theory (GSM), the damage rate is usually calculated through the rate flux laws (Lemaitre and Chaboche, 1990). However, a recent and diffuse trend in finite element modelling for

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isotropic damage assumes that the total damage at a material point depends on the total current value that a strain scalar, the equivalent strain, has ever experienced (Pijaudier-Cabot, 1995; de Borst et al., 1997; Peerlings et al., 1996; Geers, 1997; Jirásek and Patzák, 2002). The same concept has been also applied in Beatty and Krishnaswamy (2000) to describe the stress-softening behavior of rubber-like materials.

It is well known that some undesired effects can arise in modeling softening materials. For instance, the equilibrium equation loses ellipticity as the stress descends along the softening branch, and, consequently, the numerical load–displacement response is affected by mesh-dependence. As a possible remedy, a nonlocality length can be introduced by having recourse to nonlocal definitions of the equivalent strain. Dating back to the sixties, nonlocal approaches represent a currently active area of research of the finite element literature: they preserve indeed the mesh-objectivity of the numerical response (Peerlings et al., 1996; de Borst et al., 1997; Jirásek and Patzák, 2002; Polizzotto et al., 1999; Benvenuti et al., 2002; Benvenuti and Tralli, 2003). Among others, nonlocal formulations of implicit gradient type have been proposed, where the nonlocal equivalent strain is assumed as an additional independent variable (Peerlings et al., 1996; Geers, 1997). At each iteration, the equilibrium equation is solved simultaneously with an implicit gradient relationship expressing the nonlocal equivalent strain as a total strain function. These solving equations are assumed a priori, and not derived from a variational formulation.

The basic ingredients of the present formulation for softening models are that: the damage is seen as a strain function, a stepwise holonomic history is considered through an implicit backward Euler integration scheme, and nonlocal definitions of the equivalent strain are introduced. In these circumstances, an incremental strain energy function is derived, and subsequently used to construct variational formulations. The relationships between the properties of the incremental strain energy, the associativity of the damage law, and the symmetry of the tangent tensor are highlighted. Strain paths realizing the minimum increment of mechanical work are considered, and the problem of the possible lack of convexity of the incremental strain energy is also addressed.

The outline of the paper is as follows. Local definitions of the equivalent strain are first considered for simplicity. The standard backward Euler scheme is employed to integrate the damage rate evolution law along a finite strain step at a prescribed material point (Section 2). So a precise relationship between final damage and final strain is established. The boundary value problem of a body obeying an elastodamaging stress–strain law with strain-driven damage is studied (Section 2.4). The increment of the mechanical work along a holonomic step is obtained by integrating the mechanical work in the strain space. Stationarity of the increment of mechanical work is shown to provide the stress–strain relationships for elastodamaging materials (Section 3). Incremental strain energies have been recently invoked for materials undergoing finite strains and polycrystals (Ortiz and Repetto, 1999; Miehe et al., 2002), while pseudo-energies and super-potential have been derived in the past in plasticity and contact mechanics (Carter and Martin, 1976; Mistakidis and Panagiotopoulos, 1998). Like in the GSM theory, here, the increment of the mechanical work results to be the sum of two terms. The first term is path-independent, i.e. it depends only on the current value of the strain. The second term coincides with the integral over the strain path of the work performed by a stress-like term, denoted τ, depending on the strain–damage derivative. The presence of a path-dependent term agrees with previous formulations such as, for instance, Simo and Ju (1987), where the total free energy at the current instant is assumed equal to the total free energy of the initial instant plus the dissipation spent along the step. In Section 4, it is shown that, as expected, symmetry of the tangent tensor ensures associativity of the damage evolution as well as the existence of a strain potential. The definition of the equivalent strain influences the symmetry of the tangent tensor. This result agrees with Carol et al. (1994). In the case of associative damage, an explicit expression of the strain potential is obtained which is path-independent. Therefore, for this dissipative material an incremental pseudo-potential exists. In the space of the internal variables, a radial path lemma has been previously established by Nguyen (1993) and Petryk (2002). Here, radial paths in the strain space are shown to realize the minimum increment of the mechanical work (Section 5). The prerequisite is that the path-dependent term fulfils suitable conditions as, for instance, a weak symmetry condition of the tangent "stiffness" associated to the stress τ. Section 6 focuses on a one-dimensional bar subject to monotonic tensile loading and in a homogeneous strain state. In this case, any equivalent strain definition guarantees the existence of a strain potential. So, a few results worth noting can be appreciated. The first is that the strain function is globally nonconvex in the strain. The second aspect is that the presence of the path-dependent term in the increment of the mechanical work is crucial to restore the continuity of the stress–strain law at incipient damage, and it can exercise a significant influence. The cases where the damage depends on a nonlocal function of the strain field of integral or gradient type are addressed in Section 7. It is here shown that, because the nonlocality operators are linear, the previous results for a local material can be extended to the nonlocal case. The increment of the mechanical work above discussed is then used to construct multifield variational formulations (Section 7.1). Their Euler–Lagrange equations are shown to fully characterize the boundary value problem of a body of elastodamaging materials with strain-driven damage also in the presence of nonlocal constraints.

2. A computational framework for isotropic damage

2.1. Local damage

The present analysis holds for small displacements and strains and rate-independent materials. The symmetric second order strain tensor and the second order Cauchy stress tensor are here denoted by $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$, respectively, whereas the symbol $\mathbb{E}$ indicates the fourth order elasticity tensor which exhibits the typical major and minor symmetries. As usual, elastodamaging materials with isotropic damage are analyzed whose stress–strain law takes the form

$$\boldsymbol{\sigma} = (1 - \omega)\mathbb{E}\boldsymbol{\epsilon}, \tag{1}$$

where the damage scalar ω ranges from 0 for the sound material to 1 for a totally damaged material. The stress–strain law (1) aims to capture the mechanical behavior of quasi-brittle materials, and it is typically nonmonotonic: it displays an elastic path, followed by a peak and by a descending branch.

In the broad range of finite element models for isotropic elastodamaging materials, a popular class of models exists, where the damage is assumed to be governed by the equivalent strain $\bar{\epsilon}$, a scalar function of the strain, through the loading–unloading conditions

$$g = \bar{\epsilon} - \kappa \leqslant 0, \quad \dot{\omega} \geqslant 0, \quad g\dot{\omega} = 0, \tag{2}$$

where g represents the damaging function and κ the damage threshold. Because the damage evolution is not monotonic during the loading history, i.e. $\dot{\omega} \geqslant 0$, the material behavior is globally nonholonomic.

Note that, in this context, the equivalent strain definition not only influences the damage sensitivity to strain state but also practically coincides with the damage criterion itself (Jirásek and Patzák, 2002). As in plasticity, the rate of damage is ruled by

$$\dot{\omega} = \dot{\lambda}\frac{\mathrm{d}g_{\mathrm{d}}}{\mathrm{d}\Omega}, \tag{3}$$

where $\dot{\lambda} \geqslant 0$ is the damage multiplier, g_{d} is a dissipation potential not necessarily coinciding with g, and Ω denotes a variable (thermo)conjugated to ω. For instance, one of the most widely used definition of the equivalent strain is due to (Mazars and Pijaudier-Cabot, 1989):

$$\bar{\epsilon}(\boldsymbol{\epsilon}) = \sqrt{\sum_{i=1}^{3}[\epsilon_i]^2}, \tag{4}$$

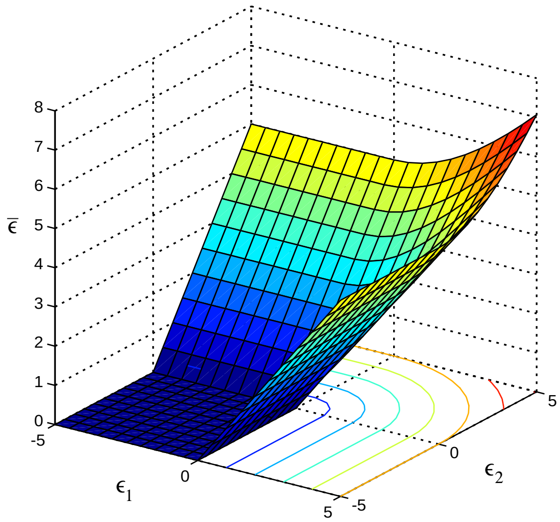

Fig. 1. Equivalent strain $\bar{\epsilon}$ corresponding to Eq. (4) in a plane-strain state.

where ϵ_i, $i = 1, 2, 3$, are the principal strains and the brackets $[\cdot]$ denote $\frac{|\cdot|+\cdot}{2}$. The surface corresponding to Eq. (4) is represented in Fig. 1 for materials in a plane strain state. Instead Fig. 3 shows the contours at $\bar{\epsilon} = 0.1\%$. It can be noted that any plane parallel to the (ϵ_1, ϵ_2)-plane does not intersect the equivalent strain surface in the negative strain domain. Because only the positive part of the principal strains appears, this definition fits well materials where damage is mainly induced by tensile rather than compressive strains. If this is not the case, alternative definitions are often considered, such as, for instance,

$$\bar{\epsilon}(\boldsymbol{\epsilon}) = \frac{r-1}{2r(1-2v)} I_1(\boldsymbol{\epsilon}) + \frac{1}{2r}\sqrt{\frac{(r-1)^2}{(1-2v)^2} I_1^2(\boldsymbol{\epsilon}) + \frac{2r}{(1+v)^2} J_2(\boldsymbol{\epsilon})}, \tag{5}$$

where v is the Poisson's ratio, and $I_1(\boldsymbol{\epsilon})$ and $J_2(\boldsymbol{\epsilon})$ are the strain tensor invariants $I_1 = tr(\boldsymbol{\epsilon})$ and $J_2 = 3tr(\boldsymbol{\epsilon}^2) - tr^2(\boldsymbol{\epsilon})$ (Peerlings et al., 1996; Geers, 1997). The scalar r denotes the ratio between the compressive and the tensile peak stresses, so that if r tends to infinity no failure due to compression can occur. Figs. 2 and 4 display, respectively, definition (5) in a plane strain state and the contour levels at $\bar{\epsilon} = 0.1\%$ for different values of the ratio r.

2.2. The incremental problem

Let us consider a body of volume V subject to a loading history during the interval of interest $[0, T]$ and restrict our attention to a point of the body. A time-like parameter t is introduced such that the interval of interest is discretized into N nonoverlapping intervals

$$[0, T] = \bigcup_{k=1}^{N} [t_{k-1}, t_k]. \tag{6}$$

At a generic initial instant t_n, the state of the body is equilibrated, consistent with the constitutive law, and characterized by the set of known variables, which are the displacement field $\mathbf{u}_n$, a strain field $\boldsymbol{\epsilon}_n$, the damage ω_n and the stress $\boldsymbol{\sigma}_n$. After application of a load step, a new equilibrated and consistent state has to

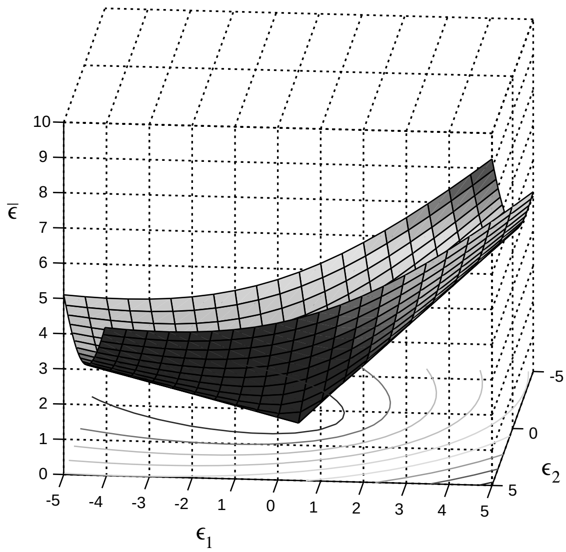

Fig. 2. Equivalent strain $\bar{\epsilon}$ corresponding to Eq. (5) in a plane-strain state.

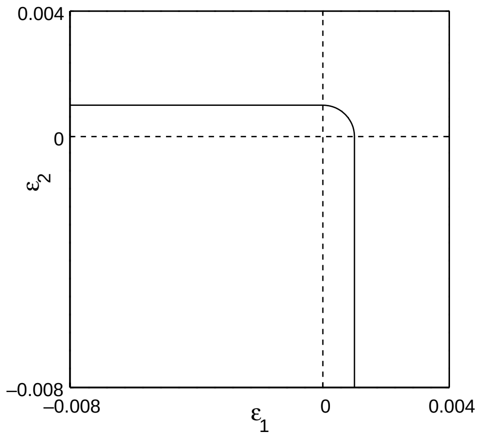

Fig. 3. Contour of the equivalent strain $\bar{\epsilon} = 0.1\%$ corresponding to Eq. (5) in plane strain; the material does not undergo damage in compression.

be calculated, characterized by the set of the updated values $(\mathbf{u}_{n+1}, \boldsymbol{\epsilon}_{n+1}, \omega_{n+1}, \boldsymbol{\sigma}_{n+1})$. Following the standard Backward Euler integration scheme, the loading function is evaluated at the final instant t_{n+1} of the step,

$$g_{n+1} = \bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}) - \kappa_{n+1}, \tag{7}$$

where $\bar{\epsilon}(\boldsymbol{\epsilon}_{n+1})$ is the equivalent strain evaluated at the final instant t_{n+1} (Simo and Hughes, 1997). In Eq. (7), the damage threshold κ_{n+1} represents the highest value the equivalent strain has ever reached during the entire loading history

$$\kappa_{n+1} = \sup_{t_i \in [t_0, t_{n+1}]} \bar{\epsilon}(\boldsymbol{\epsilon}_{t_i}). \tag{8}$$

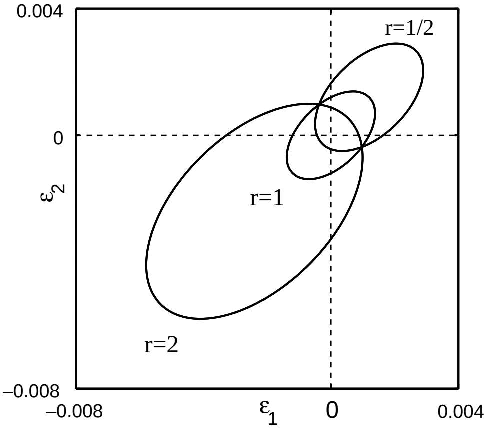

Fig. 4. Contours of the equivalent strain $\bar{\epsilon} = 0.1\%$ corresponding to Eq. (5) setting $\nu = 0.2$ and varying r in plane-strain; the material is more (resp. less) resistant in compression than in traction for $r > 1$ (resp. $r < 1$), but damageable for both loadings.

The loading–unloading conditions (2) become

$$g_{n+1} = \bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}) - \kappa_{n+1}, \quad \Delta\omega \geqslant 0, \quad g_{n+1}\Delta\omega = 0, \tag{9}$$

where $\Delta\omega = \omega_{n+1} - \omega_n$.

As shown in Florez-Lopez et al. (1994), if the damage process is active and $\bar{\epsilon}$ and κ are strictly increasing real valued functions of the strain, then the rate law (3) can be integrated over time to obtain a damage–strain evolution law of the integrated form

$$\omega_{n+1} = \begin{cases} 0 & \text{if } \kappa_{n+1} \leqslant \kappa_{\mathrm{i}}, \\ \omega(\kappa_{n+1}) & \text{if } \kappa_{\mathrm{i}} \leqslant \kappa_{n+1} \leqslant \kappa_{\mathrm{f}}, \\ 1 & \text{if } \kappa_{n+1} \geqslant \kappa_{\mathrm{f}}. \end{cases} \tag{10}$$

Integrated laws of evolution of the internal variables have been obtained for both damaging materials, e.g. in Lemaitre and Chaboche (1990) by a step-by-step integration along monotonic loading, and plastic materials, e.g. in Ortiz and Repetto (1999) by integration of the evolution equation of the internal variable along "minimizing" deformation paths. In Eq. (10), κ_{i} and κ_{f} denote an initial and a final damage threshold. Nevertheless, as argued in Carol et al. (1994), one can think of damage rules that cannot be obtained by integration of a rate rule. This is very frequent in the FE-oriented literature. For instance, in Peerlings et al. (1996) and Geers (1997), the exponential law

$$\omega_{n+1} = \begin{cases} 0 & \text{if } \kappa_{n+1} < \kappa_{\mathrm{i}}, \\ 1 - \left(\frac{\kappa_{\mathrm{i}}}{\kappa_{n+1}}\right)^{\beta} \left(\frac{\kappa_{\mathrm{f}} - \kappa_{n+1}}{\kappa_{\mathrm{f}} - \kappa_{\mathrm{i}}}\right)^{\alpha} & \text{if } \kappa_{\mathrm{i}} < \kappa_{n+1} < \kappa_{\mathrm{f}}, \\ 1 & \text{if } \kappa_{n+1} > \kappa_{\mathrm{f}} \end{cases} \tag{11}$$

has been used together with the asymptotic damage evolution law

$$\omega_{n+1} = \begin{cases} 0 & \text{if } \kappa_{n+1} < \kappa_{\mathrm{i}}, \\ 1 - \frac{\kappa_{\mathrm{i}}}{\kappa_{n+1}} \left(1 - \alpha + \alpha \mathrm{e}^{-\beta(\kappa_{n+1} - \kappa_{\mathrm{i}})}\right) & \text{otherwise}, \end{cases} \tag{12}$$

where the exponents α and β make it possible to get a wide set of constitutive laws.

Some preliminary comments on the above damage evolution law are at this stage necessary. For instance, as soon as the damage threshold κ_{n+1} defined in Eq. (8) equals κ_{i}, the damage ω_{n+1} is a strictly increasing function of κ_{n+1}. This aspect emerges from Fig. 5, where the same initial damage threshold has

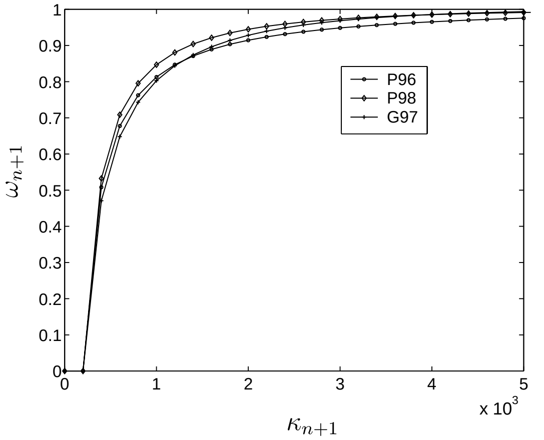

Fig. 5. Comparison of damage laws versus the threshold using $\kappa_i = 0.0002$; in Eq. (12) (P96) $\alpha = 0.96$, $\beta = 350$ (P98), $\kappa_f = 0.0125$ in Eq. (11) (G97) $\alpha = 5$, $\beta = 0.8$, $\kappa_f = 0.0125$.

been assumed for all the models. Moreover, the influence of the equivalent strain definitions on the stress–strain laws should be illustrated to the sake of completeness. For this purpose, a one-dimensional tensile bar has been considered under a homogeneous strain regime (see the captions of the figures below for what concerns with the constitutive parameters). In particular, the one-dimensional stress–strain laws corresponding to the damage laws of Fig. 5 are represented in Fig. 6 assuming a monotonic loading. As can be deduced from Fig. 7, the equivalent strain definition (4) leads to a one-dimensional stress–strain law that is elastic in compression. Instead, the stress–strain law corresponding to the equivalent strain definition (5) exhibits a softening behavior in both tension and compression and is sensitive to the Poisson coefficient (Fig. 8). Another relevant point is that the damage ω_{n+1} (10) is a continuous function of the threshold κ_{n+1}, whereas its first derivative is discontinuous at incipient damage $\kappa_{n+1} = \kappa_i$, because, there, the left and right derivatives of ω_{n+1} do not coincide. This can be observed, for instance, assuming the exponential damage evolution (11), so that

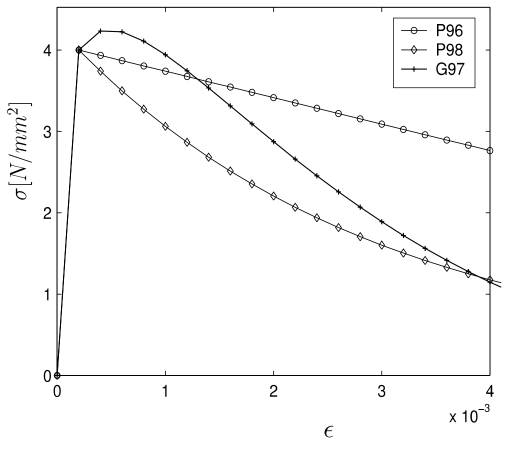

Fig. 6. One-dimensional stress–strain laws corresponding to the data of Fig. 5 for a material with Young modulus $E = 20{,}000$ N/mm^2 and a monotonic loading.

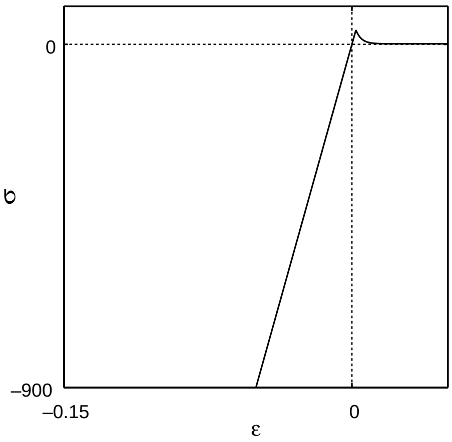

Fig. 7. One-dimensional stress–strain law using the equivalent strain (4) and the damage law (12) with $r = 10$, $\alpha = 0.96$, $\beta = 350$ and a monotonic loading.

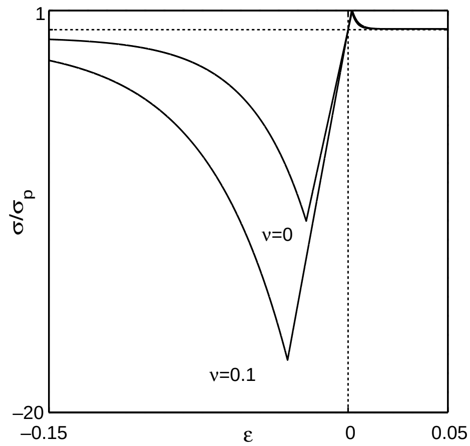

Fig. 8. Normalized one-dimensional stress–strain law (5) using the damage law (12), with $r = 10$, $\alpha = 0.96$, $\beta = 350$, $\nu = 0$ or $\nu = 0.1$ and with a monotonic loading.

$$\left.\frac{\mathrm{d}\omega}{\mathrm{d}\kappa}\right|_{n+1} \equiv \frac{\mathrm{d}\omega_{n+1}}{\mathrm{d}\kappa_{n+1}} = \begin{cases} 0 & \text{if } \kappa_{n+1} < \kappa_{\mathrm{i}}, \\ (1-\omega_{n+1})\left(\frac{\beta}{\epsilon}+\frac{\alpha}{\kappa_{\mathrm{f}}-\kappa_{n+1}}\right) & \text{if } \kappa_{\mathrm{i}} \leqslant \kappa_{n+1} \leqslant \kappa_{\mathrm{f}}, \\ 0 & \text{if } \kappa_{n+1} > \kappa_{\mathrm{f}}. \end{cases} \tag{13}$$

As shown in Fig. 9 for fixed $\alpha = 1$ and increasing values of β, the discontinuity of the damage derivative at incipient damage is strongly influenced by the choice of the model parameters. Moreover, the right derivative can reach values several orders of magnitude larger than the left derivative as discussed in Section 6. The presence of the discontinuity of the damage derivative at incipient damage has been previously pointed out in Frèmond and Nedjar (1996). For instance, in this latter model, this problem is overcome by setting the first derivative of the damage at $\kappa = \kappa_{\mathrm{i}}$ equal to the right one. In the following, functions κ_{n+1} and ω_{n+1} are required to be sufficiently regular for their directional derivatives to be continuous almost everywhere.

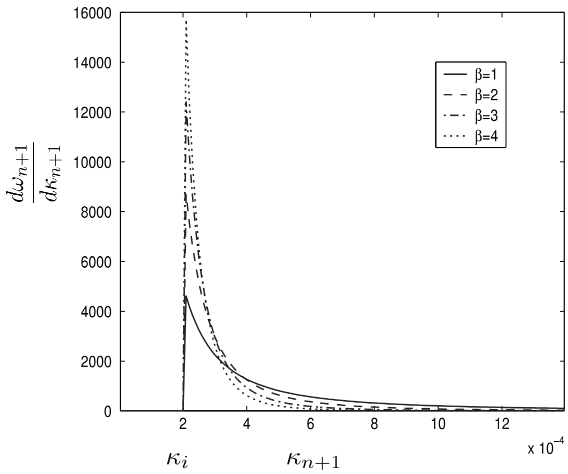

Fig. 9. Derivative of the damage with respect to the threshold versus the threshold using the damage law (12) with $\alpha = 1$ and increasing β.

2.3. Incremental damage integration

Two circumstances may occur, consisting either in an elastic step, where damage does not increase, or in a damaging step, where damage increases.

- *Elastic step:* Suppose that

$$g_{n+1} = \bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}) - \kappa_n \leqslant 0. \tag{14}$$

The old damage threshold κ_n is replaced by the current one κ_{n+1}, but they actually coincide:

$$\kappa_{n+1} = \kappa_n. \tag{15}$$

According to Eq. (10), ω_{n+1} is a strictly increasing function of κ_{n+1}. Therefore, the damage does not increase,

$$\omega_{n+1} = \omega_n. \tag{16}$$

- *Damaging step:* Otherwise, if

$$g_{n+1} = \bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}) - \kappa_n > 0, \tag{17}$$

then the updated threshold κ_{n+1} calculates as

$$\kappa_{n+1} = \bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}). \tag{18}$$

Because $\bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}) > \kappa_n$, the damage increases, i.e. $\omega_{n+1} > \omega_n$, and

$$\omega_{n+1} = \omega(\kappa_{n+1}) = \omega(\bar{\epsilon}(\boldsymbol{\epsilon}_{n+1})). \tag{19}$$

In both the above circumstances, the loading–unloading conditions are restored at the final instant. It can be noted that, here, unlike in plasticity, the standard predictor–corrector format condenses into a unique phase, because the loading function involves only scalar functions of the strain.

Consequently, the damage threshold κ_{n+1} is a *nondecreasing* function of the strain of the type:

$$\boxed{\kappa_{n+1} = \max\{\kappa_n, \bar{\epsilon}(\boldsymbol{\epsilon}_{n+1})\},} \tag{20}$$

where $\kappa_{n+1} = \kappa_n$ in an elastic step, and $\kappa_{n+1} = \bar{\epsilon}(\boldsymbol{\epsilon}_{n+1})$ in a damaging step. Because the damage is in turn strictly increasing with κ_{n+1}, it is a *nondecreasing* function of the strain:

$$\boxed{\omega_{n+1} = \max\{\omega_n, \omega(\bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}))\}.} \tag{21}$$

As expected, the damage evolution is path-dependent and depends on the strain-history through ω_n. By replacing the strain–damage evolution law ω_{n+1} (21) in the stress–strain relationship (1), the stress–strain law at the instant t_{n+1} writes

$$\boldsymbol{\sigma}_{n+1} = (1 - \omega_{n+1})\mathbb{E}\boldsymbol{\epsilon}_{n+1}. \tag{22}$$

2.4. The incremental boundary value problem

A body is given under volume forces $\mathbf{b}_{n+1}$ applied on the volume V with boundaries ∂V. For simplicity, $\mathbf{u}$ vanishes on ∂V and no other boundary conditions are considered. Let us assume that a strain function $\Psi(\boldsymbol{\epsilon}_{n+1})$ exists, such that the extended Hu–Washizu formulation

$$\mathscr{F}(\mathbf{u}_{n+1}, \boldsymbol{\epsilon}_{n+1}, \boldsymbol{\sigma}_{n+1}) \equiv \int_V \Psi(\boldsymbol{\epsilon}_{n+1}) + \boldsymbol{\sigma}_{n+1} \cdot (\nabla^s \mathbf{u}_{n+1} - \boldsymbol{\epsilon}_{n+1}) - \mathbf{b}_{n+1} \cdot \mathbf{u}_{n+1}\, \mathrm{d}V \tag{23}$$

may be constructed, where ∇^s denotes the symmetric part of the strain tensor. For brevity, in the following, it is set $\Psi(\boldsymbol{\epsilon}_{n+1}) \equiv \Psi_{n+1}$. The virtual variations of $\mathbf{u}$ and $\boldsymbol{\sigma}$ are assumed sufficiently regular, i.e. $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\sigma}}$ are such that any component $\hat{u}_i$ and $\hat{\sigma}_{ij}$ are of summable square (Simo and Hughes, 1997). The admissible virtual variations of the strain $\hat{\boldsymbol{\epsilon}}$ satisfy the homogeneous boundary condition and are sufficiently regular for the subsequent calculations to make sense. The first variations $\delta\mathscr{F}$ of the function (23)

$$\delta\mathscr{F}(\mathbf{u}_{n+1}, \boldsymbol{\epsilon}_{n+1}, \boldsymbol{\sigma}_{n+1}; \hat{\mathbf{u}}) = \int_V \boldsymbol{\sigma}_{n+1} \cdot \nabla^s \hat{\mathbf{u}} - \mathbf{b}_{n+1} \cdot \hat{\mathbf{u}}\, \mathrm{d}V, \tag{24a}$$

$$\delta\mathscr{F}(\mathbf{u}_{n+1}, \boldsymbol{\epsilon}_{n+1}, \boldsymbol{\sigma}_{n+1}; \hat{\boldsymbol{\epsilon}}) = \int_V \left(\boldsymbol{\sigma}_{n+1} - \left.\frac{\mathrm{d}\Psi}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1}\right) \cdot \hat{\boldsymbol{\epsilon}}\, \mathrm{d}V, \tag{24b}$$

$$\delta\mathscr{F}(\mathbf{u}_{n+1}, \boldsymbol{\epsilon}_{n+1}, \boldsymbol{\sigma}_{n+1}; \hat{\boldsymbol{\sigma}}) = \int_V (\nabla^s \mathbf{u}_{n+1} - \boldsymbol{\epsilon}_{n+1}) \cdot \hat{\boldsymbol{\sigma}}\, \mathrm{d}V \tag{24c}$$

are equated to zero for any set of admissible variations $(\hat{\mathbf{u}}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\epsilon}})$. For brevity of notation, here $\frac{\mathrm{d}\Psi}{\mathrm{d}\boldsymbol{\epsilon}}|_{n+1}$ stands for $\frac{\mathrm{d}\Psi_{n+1}}{\mathrm{d}\boldsymbol{\epsilon}_{n+1}}$. The corresponding strong form equations lead to formulate the problem below:

P1. Find $\mathscr{F}(\mathbf{u}_{n+1}, \boldsymbol{\epsilon}_{n+1}, \boldsymbol{\sigma}_{n+1})$ s.t. solving $\delta\mathscr{F} = \mathbf{0}$ for any $(\hat{\mathbf{u}}, \hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{\sigma}})$ leads to

$$\mathrm{div}\, \boldsymbol{\sigma}_{n+1} + \mathbf{b}_{n+1} = \mathbf{0}, \quad \text{in } V, \tag{25a}$$

$$\boldsymbol{\sigma}_{n+1} = \left.\frac{\mathrm{d}\Psi}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1}, \quad \text{in } V, \tag{25b}$$

$$\nabla^s \mathbf{u}_{n+1} = \boldsymbol{\epsilon}_{n+1}, \quad \text{in } V, \tag{25c}$$

where $\boldsymbol{\sigma}_{n+1} = (1 - \omega_{n+1})\mathbb{E}\boldsymbol{\epsilon}_{n+1}$ and $\omega_{n+1} = \max\{\omega_n, \omega(\bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}))\}$.

Eq. (25) are the classical solving equations of boundary value problems for elastodamaging materials obeying the constitutive model governed by Eq. (21).

3. The increment of mechanical work

In this section, the mechanical work is integrated along a strain step. This provides the increment of the mechanical work, which results to be a strain function consisting of a path-independent and a path-dependent term. Analogously, in the model based on strain driven damage (Simo and Ju, 1987), the strain energy at the current instant has been assumed equal to the total free energy of the initial instant plus the dissipation spent along the step. In the stress space, Ortiz has formulated the energy potential describing damage in concrete as the sum of the elastic part and the energy amount required to open microcracks (Ortiz, 1985). However, the present strain function is not an elastic strain potential in a strict sense, because the material behavior is nonlinear. Nevertheless, several (pseudo-)elastic theories have been developed for nonlinear materials: for instance, in the past, by Carter and Martin (1976) in hardening plasticity, more recently, by Ortiz and Repetto (1999) in crystal plasticity, and by Miehe et al. (2002) in a homogenization analysis of inelastic materials and for polycrystals in finite elasticity. By integrating the constitutive relationships along deformation histories which minimize the work of deformation, these latter Authors have shown that the resulting stress–strain relations take a pseudo-elastic form, with the work of deformation itself supplying the appropriate strain energy potential (Ortiz and Repetto, 1999). Moreover, within the nonsmooth mechanics framework, nonmonotonic stress–strain laws have been related by Panagiotopoulos and coworkers to the existence of (nonconvex) strain super-potentials (Mistakidis and Panagiotopoulos, 1998). In other contexts, biological tissues have been modelled with suitable pseudo-strain energy functions, although they cannot have a strain energy in the thermodynamic sense (Fung, 1993). Analogously, rubberlike materials undergo irreversible changes of the mechanical properties after unloading. Nevertheless, their behavior has been described through pseudo-energy functions of the finite strain tensor, where the total damage can be related to the finite strain itself (Ogden, 2000).

3.1. Integration of the mechanical work

The evolution of the structural response is usually analyzed as a sequence of incremental problems, each concerning a configuration change from a previously known state due to a finite increment of load step. Each nonholonomic problem can then be transformed through an implicit backward difference integration scheme into a stepwise holonomic problem (Comi et al., 1992; Simo and Hughes, 1997; Tin-Loi and Xia, 2001). Let

$$\mathscr{H} \equiv \left\{ \boldsymbol{\epsilon}(t),\ \boldsymbol{\epsilon}(t_n) = \boldsymbol{\epsilon}_n,\ \boldsymbol{\epsilon}(t_{n+1}) = \boldsymbol{\epsilon}_{n+1},\ \text{such that either } \dot{\omega} = 0 \text{ or } \dot{\omega} > 0 \right\} \tag{26}$$

identify the set of all strain paths from $\boldsymbol{\epsilon}_n$ to $\boldsymbol{\epsilon}_{n+1}$ for t in $[t_n, t_{n+1}]$, along which the damage rate is strictly monotonic. The attribute of holonomic will here connote strain paths in $\mathscr{H}$. In this spirit, the material behavior is assumed as stepwise holonomic. By Eq. (25b) and by the assumption that the damage evolves according to (21), the strain function Ψ_{n+1} is required to be such that its derivative with respect to $\boldsymbol{\epsilon}_{n+1}$ coincides with $\boldsymbol{\sigma}_{n+1}$ (22). Therefore, for the purpose of solving Problem P1, one can restrict oneself to the study of the problem below:

P2. Find Ψ_{n+1} such that:

$$\boldsymbol{\sigma}_{n+1} = \left.\frac{\mathrm{d}\Psi}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1}, \tag{27a}$$

$$\boldsymbol{\sigma}_{n+1} = (1-\omega_{n+1})\mathbb{E}\boldsymbol{\epsilon}_{n+1}, \tag{27b}$$

where $\omega_{n+1} = \max\{\omega_n, \omega(\bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}))\}$.

If summable, the increment of the mechanical work can be integrated

$$\Delta\Psi = \int_{t_n}^{t_{n+1}} \boldsymbol{\sigma}(t)\cdot\dot{\boldsymbol{\epsilon}}(t)\,\mathrm{d}t \tag{28}$$

along a strain path $\boldsymbol{\epsilon}(t) \in \mathscr{H}$ where $\Delta\Psi \equiv \Psi_{n+1} - \Psi_n$. Note that by integrating the mechanical work, only the regular part of the increment of the mechanical work can be recovered, whereas its singular and jump parts are lost (Kolmogorov and Fomin, 1970). Integration by parts of the right-hand-side of Eq. (28) provides

$$\Delta\Psi = (1-\omega_{n+1})Y_{n+1} - (1-\omega_n)Y_n - \int_{t_n}^{t_{n+1}} Y(t)\frac{\mathrm{d}}{\mathrm{d}t}(1-\omega(t))\,\mathrm{d}t, \tag{29}$$

where, for brevity,

$$Y(t) \equiv \frac{1}{2}\boldsymbol{\epsilon}(t)\cdot\mathbb{E}\boldsymbol{\epsilon}(t), \qquad Y_n \equiv \frac{1}{2}\boldsymbol{\epsilon}_n\cdot\mathbb{E}\boldsymbol{\epsilon}_n, \qquad Y_{n+1} \equiv \frac{1}{2}\boldsymbol{\epsilon}_{n+1}\cdot\mathbb{E}\boldsymbol{\epsilon}_{n+1}. \tag{30}$$

After simple calculations, Eq. (29) becomes

$$\Delta\Psi = (1-\omega_{n+1})Y_{n+1} - (1-\omega_n)Y_n + \int_{t_n}^{t_{n+1}} Y(t)\dot{\omega}(t)\,\mathrm{d}t. \tag{31}$$

Because $\dot{\omega} = \frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\cdot\dot{\boldsymbol{\epsilon}}$ by the chain rule, the integral term can be written

$$\int_{t_n}^{t_{n+1}} Y(t)\dot{\omega}(t)\,\mathrm{d}t = \int_{t_n}^{t_{n+1}} Y(t)\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\cdot\dot{\boldsymbol{\epsilon}}(t)\,\mathrm{d}t. \tag{32}$$

Replacing the definition of Y (30a) in Eq. (31), the increment of the mechanical work (31) becomes

$$\Delta\Psi = \Delta\Phi + \int_{t_n}^{t_{n+1}} \frac{1}{2}\left(\left.\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\right|_t \otimes \mathbb{E}\boldsymbol{\epsilon}(t)\right)\boldsymbol{\epsilon}(t)\cdot\dot{\boldsymbol{\epsilon}}(t)\,\mathrm{d}t, \tag{33}$$

where

$$\Delta\Phi \equiv \Phi_{n+1} - \Phi_n, \quad \Phi_{n+1} \equiv (1-\omega_{n+1})Y_{n+1}, \quad \Phi_n \equiv (1-\omega_n)Y_n. \tag{34}$$

Eq. (33) makes it possible to identify the stress $\boldsymbol{\tau}(t)$ conjugated to the strain increment $\dot{\boldsymbol{\epsilon}}(t)$ along the considered paths and having constitutive equation

$$\boldsymbol{\tau} = \begin{cases} \mathbf{0} & \text{if } \dot{\omega} = 0, \\ \frac{1}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}} \otimes \mathbb{E}\boldsymbol{\epsilon}\right)\boldsymbol{\epsilon} & \text{if } \dot{\omega} > 0, \end{cases} \tag{35}$$

where the t-dependence has been omitted for simplicity and the holonomy assumption along the strain step has been used. In the GSM framework, the change of the mechanical work is frequently split into the change of free energy and the dissipation. Analogously, the work increment (33) can be written as the sum of two contributions, i.e.

$$\Delta\Psi = \Delta\Phi + \Delta\Gamma, \tag{36}$$

where $\Delta\Phi$ is path-independent, whereas

$$\Delta\Gamma \equiv \int_{t_n}^{t_{n+1}} \boldsymbol{\tau}(t)\cdot\dot{\boldsymbol{\epsilon}}(t)\,\mathrm{d}t \tag{37}$$

is path-dependent.

Let us focus on the integral term of Eq. (32). Its argument is summable on the interval of integration because Y is a positive definite quadratic form, and $\dot{\omega}$ is bounded. Thus, the fundamental lemma of the integral calculus can be applied (Kolmogorov and Fomin, 1970)

$$\frac{\mathrm{d}}{\mathrm{d}t_{n+1}}\int_{t_n}^{t_{n+1}} Y(t)\dot{\omega}(t)\,\mathrm{d}t = Y_{n+1}\dot{\omega}(t_{n+1}). \tag{38}$$

Let us write explicitly the strain functional Ψ_{n+1} associated to the increment of mechanical work (36)

$$\Psi_{n+1} = \Phi_{n+1} - \Phi_n + \int_{\boldsymbol{\epsilon}_n}^{\boldsymbol{\epsilon}_{n+1}} Y(\boldsymbol{\epsilon})\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\cdot\mathrm{d}\boldsymbol{\epsilon} + \Psi_n, \tag{39}$$

where interchangeability of the time-like parameter with the strain has been assumed and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(t)$ varies from $\boldsymbol{\epsilon}_n$ to $\boldsymbol{\epsilon}_{n+1}$.This is possible if $\boldsymbol{\epsilon}$ in $\mathscr{H}$ (26) is an invertible function of t. By multiplying by $\frac{\mathrm{d}t}{\mathrm{d}\boldsymbol{\epsilon}}|_{n+1}$ both sides of Eq. (38), one gets:

$$\left.\frac{\mathrm{d}\Delta\Gamma}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1} = Y_{n+1}\left.\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1}. \tag{40}$$

Therefore, by differentiating the strain functional Ψ_{n+1} with respect to $\boldsymbol{\epsilon}_{n+1}$

$$\left.\frac{\mathrm{d}\Psi}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1} = (1-\omega_{n+1})\mathbb{E}\boldsymbol{\epsilon}_{n+1} - Y_{n+1}\left.\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1} + Y_{n+1}\left.\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1} \tag{41}$$

leads to the stress–strain law (1). So Ψ_{n+1} turns out to be the strain energy associated to the stress $\boldsymbol{\sigma}_{n+1}$ along those strain paths in $\mathscr{H}$ such that the strain-time dependence is invertible. That solves Problem P2. Note that the tangent tensor $\mathbb{H}_{n+1}$ can be calculated as

$$\mathbb{H}_{n+1} = \left.\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1} = (1-\omega_{n+1})\mathbb{E} - \mathbb{E}\boldsymbol{\epsilon}_{n+1}\otimes\left.\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1}. \tag{42}$$

The positive definiteness of the tangent elastoplastic tensor ensures stability of the material. However, in the case of softening stress–strain laws, the tangent tensor $\mathbb{H}_{n+1}$ is not expected to be positive definite. Explicit expressions of the tangent tensor analogous to Eq. (42) were derived in Lubarda and Krajcinovic (1995) starting from rate-type constitutive relationships and in the more general case of anisotropic damage for quasi brittle materials.

4. Potential existence and associativity

A directionally differentiable functional $\Psi(\boldsymbol{\epsilon})$ is the strain potential of $\boldsymbol{\sigma}(\boldsymbol{\epsilon})$ if and only if $\boldsymbol{\sigma}(\boldsymbol{\epsilon}) = \nabla\Psi(\boldsymbol{\epsilon})$, where $\nabla\Psi(\boldsymbol{\epsilon})$ denotes the gradient of Ψ at $\boldsymbol{\epsilon}$. Existence of a strain potential and associativity of the damage evolution can be deduced as a consequence of the major symmetry property of the tangent tensor (Hill, 1959). The problem whether the strain energy Ψ associated to the strain $\boldsymbol{\sigma}$ represents the strain potential of $\boldsymbol{\sigma}$ can be studied as an inverse problem in the calculus of variations. According to Oden and Reddy (1976), a necessary and sufficient condition for $\boldsymbol{\sigma}$ to be potential is that the bilinear functional $\delta\boldsymbol{\sigma}(\boldsymbol{\epsilon},\boldsymbol{\epsilon}_1)\cdot\boldsymbol{\epsilon}_2$, where $\delta\boldsymbol{\sigma}$ denotes the directional differential $\delta\boldsymbol{\sigma}(\boldsymbol{\epsilon};\boldsymbol{\epsilon}_1)$ of $\boldsymbol{\sigma}$ at $\boldsymbol{\epsilon}$ in the direction $\boldsymbol{\epsilon}_1$, is symmetric with respect to $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ for each $\boldsymbol{\epsilon}$:

$$\delta\boldsymbol{\sigma}(\boldsymbol{\epsilon};\boldsymbol{\epsilon}_1)\cdot\boldsymbol{\epsilon}_2 = \delta\boldsymbol{\sigma}(\boldsymbol{\epsilon};\boldsymbol{\epsilon}_2)\cdot\boldsymbol{\epsilon}_1. \tag{43}$$

If the symmetry condition (43) holds, then the strain functional Ψ_{n+1} Eq. (39) is also an incremental potential for the stress $\boldsymbol{\sigma}_{n+1}$ (22) and it is independent of the chosen path of integration (Lubarda and Krajcinovic, 1995). The directional differential $\delta\boldsymbol{\sigma}(\boldsymbol{\epsilon};\boldsymbol{\epsilon}_1)$ calculates as

$$\delta\boldsymbol{\sigma}(\boldsymbol{\epsilon};\boldsymbol{\epsilon}_1) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\sigma}(\boldsymbol{\epsilon}+t\boldsymbol{\epsilon}_1))\bigg|_{t=0} = -\left(\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\cdot\boldsymbol{\epsilon}_1\right)\mathbb{E}\boldsymbol{\epsilon} + (1-\omega(\boldsymbol{\epsilon}+t\boldsymbol{\epsilon}_1))\bigg|_{t=0}\mathbb{E}\boldsymbol{\epsilon}_1. \tag{44}$$

Therefore, by the symmetry of $\mathbb{E}$

$$\delta\boldsymbol{\sigma}(\boldsymbol{\epsilon};\boldsymbol{\epsilon}_{\mathrm{i}})\cdot\boldsymbol{\epsilon}_j = -\left(\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\cdot\boldsymbol{\epsilon}_{\mathrm{i}}\right)\boldsymbol{\epsilon}_j\cdot\mathbb{E}\boldsymbol{\epsilon} + (1-\omega(\boldsymbol{\epsilon}))\boldsymbol{\epsilon}_j\cdot\mathbb{E}\boldsymbol{\epsilon}_{\mathrm{i}}, \tag{45}$$

where $i \neq j$, $i,j = 1,2$. By Eq. (45) and by the symmetry of $\mathbb{E}$, Eq. (43) holds if an only if

$$\left(\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\cdot\boldsymbol{\epsilon}_1\right)(\boldsymbol{\epsilon}\cdot\mathbb{E}\boldsymbol{\epsilon}_2) = \left(\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\cdot\boldsymbol{\epsilon}_2\right)(\boldsymbol{\epsilon}\cdot\mathbb{E}\boldsymbol{\epsilon}_1) \tag{46}$$

for any $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$. Eq. (46) is equivalent to state that the fourth order tensor $\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\otimes\mathbb{E}\boldsymbol{\epsilon}$ exhibits major symmetry:

$$\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\otimes\mathbb{E}\boldsymbol{\epsilon} = \mathbb{E}\boldsymbol{\epsilon}\otimes\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}. \tag{47}$$

Eq. (47) implies the symmetry of both the stress tensor $\boldsymbol{\tau}$ (36) and the tangent modulus of the constitutive stress–strain law (22) $\mathbb{H}_{n+1}$. The major symmetry of the algorithmic tangent tensor $\mathbb{H}_{n+1}$ plays an important role in the choice of the equation solver in numerical computations of boundary value problems. In particular, the symmetry condition (47) holds when:

- the strain tensor reduces to a scalar;
- the second order tensors $\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}$ and $\mathbb{E}\boldsymbol{\epsilon}$ are coaxial.

The first assertion is trivial, the second circumstance is commented in the section below.

4.1. Associative damage

Coaxiality of the tensors $\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}$ and $\mathbb{E}\boldsymbol{\epsilon}$ means here that

$$\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}} = \sum_{i=1}^{6}\alpha_{\mathrm{i}}\mathbf{e}_{\mathrm{i}}, \qquad \mathbb{E}\boldsymbol{\epsilon} = \sum_{i=1}^{6}\beta_{\mathrm{i}}\mathbf{e}_{\mathrm{i}}, \tag{48}$$

where $\mathbf{e}_{\mathrm{i}}$ is an orthonormal basis of $\mathbb{S}$, the space of the symmetric second order tensors. In a standard GSM framework, Carol et al. (1994) have shown that the above coaxiality condition (47) implies associativity of the damage law in the strain space. Damage flux laws of associative type occur when the increment of the damage is proportional to the gradient of the damage surface. This simplification is generally not acceptable in the presence of materials with a highly dilatant behavior such as geomaterials. In the latter case, nonassociative damage flux laws should be considered and the results reported in this section do not hold.

The following equivalent strain definitions:

- $\bar{\epsilon} = mY$ (Ju, 1989; de Borst et al., 1997),
- $\bar{\epsilon} = m\sqrt{2Y}$ (Simo and Ju, 1987),

where m is a positive scalar, satisfy the coaxiality (symmetry) condition (47) and thus imply associative damage. Other examples of models where the equivalent strain definition fulfils the coaxiality condition (47) can be found in Benallal et al. (1986) and Neilsen and Schreyer (1992). As pointed out in Carol et al. (1994), the equivalent strain definitions of Eqs. (4) and (5) do not satisfy the symmetry condition (47). Let the coaxiality condition be satisfied by assuming

$$\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}} = \alpha\mathbb{E}\boldsymbol{\epsilon}. \tag{49}$$

In Eq. (49), according to the hypothesis of holonomic behavior within the strain-step, α is any positive real number, such that α vanishes when $\dot{\omega}$ vanishes and is greater than 0 when $\dot{\omega}$ does not vanish. Eq. (49) ensures that the damage evolution is of associative type. Then, the integral term $\Delta\Gamma$ (37)

$$\int_{\boldsymbol{\epsilon}_n}^{\boldsymbol{\epsilon}_{n+1}} \alpha Y \mathbb{E}\boldsymbol{\epsilon}\,\mathrm{d}\boldsymbol{\epsilon} = \frac{1}{2}\alpha(Y_{n+1}^2 - Y_n) \tag{50}$$

results to be path-independent. In synthesis, when the above coaxiality condition (49) holds, the increment of the mechanical work along a strain path is path-independent, but depends only on the initial and final value of the strain. Moreover, the increment can be written as

$$\Delta\Psi = \begin{cases} \Delta\Phi + \frac{1}{2}\alpha(Y_{n+1}^2 - Y_n^2) & \text{if } \dot{\omega} > 0, \\ \Delta\Phi & \text{if } \dot{\omega} = 0 \end{cases} \tag{51}$$

along holonomic damaging steps. By the hypothesis (49),

$$\sigma_{n+1} = \left.\frac{\mathrm{d}\Psi}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1} = (1-\omega_{n+1})\left.\frac{\mathrm{d}Y}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1} - \left.\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1} Y_{n+1} + Y_{n+1}\left.\frac{\alpha\,\mathrm{d}Y}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1} = (1-\omega_{n+1})\mathbb{E}\boldsymbol{\epsilon}_{n+1}, \tag{52}$$

so that $\Delta\Psi$ (51) is the potential of the stress σ_{n+1}.

5. Strain radial paths as minimizers of $\Delta\Psi$

In *associative hardening* elastoplasticity, the standard predictor–corrector scheme can be deduced from a convex minimization problem where the positive definiteness of the tangent modulus and the convexity of the complementarity energy are assumed (Simo and Hughes, 1997). On the contrary, this result cannot be extended to nonassociative hardening plasticity and softening plasticity. In the present class of softening elastodamaging materials, positive definiteness of the tangent tensor is not ensured and the strain energy can be nonconvex, so that it is not simple to find an explicit expression of the complementary energy. However, here, some results are presented which do not require positive definiteness of the tangent modulus and global convexity of the change of mechanical work. In particular, it is shown that the path dependent term $\Delta\Gamma$ can be minimized along a radial strain path provided that suitable additional conditions are satisfied. This result agrees with the radial path lemma proposed by Nguyen (1993) in the space of the internal variables and later extended by Petryk (2002).

An indirect strain path is considered starting from $\boldsymbol{\epsilon}_n$ and leading to the final strain increment $\Delta\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{n+1} - \boldsymbol{\epsilon}_n$. Following an analogous procedure in Petryk (2002), let us linearize the path-dependent term around $\boldsymbol{\epsilon}_n$ as follows:

$$\boldsymbol{\tau} \cdot \boldsymbol{\epsilon} = \boldsymbol{\tau}_n \cdot \dot{\boldsymbol{\epsilon}} + \Delta\boldsymbol{\epsilon} \cdot \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}} + \mathrm{o}(\|\Delta\boldsymbol{\epsilon}\|^2), \tag{53}$$

where $\boldsymbol{\tau}_n = \boldsymbol{\tau}(\boldsymbol{\epsilon}_n)$. So the change $\Delta\Gamma$ can be split into a first-order and a second-order contribution

$$\Delta\Gamma = \Delta_1\Gamma + \Delta_2\Gamma + \mathrm{o}(\|\Delta t\|^2), \tag{54}$$

where

$$\Delta_1\Gamma = \int_{t_n}^{t_{n+1}} \boldsymbol{\tau}_n \cdot \dot{\boldsymbol{\epsilon}}(t)\,\mathrm{d}t, \qquad \Delta_2\Gamma = \int_{t_n}^{t_{n+1}} \Delta\boldsymbol{\epsilon} \cdot \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(t)\,\mathrm{d}t. \tag{55}$$

Attention is now focused on the argument of the second order term $\Delta_2\Gamma$

$$\begin{aligned} 2\Delta\boldsymbol{\epsilon} \cdot \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(t) &= \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(t) \cdot \Delta\boldsymbol{\epsilon} + \int_{t_n}^{t} \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(t) \cdot \dot{\boldsymbol{\epsilon}}(s)\,\mathrm{d}s \\ &= \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(t) \cdot \Delta\boldsymbol{\epsilon} + \int_{t_n}^{t} \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(s) \cdot \dot{\boldsymbol{\epsilon}}(t)\,\mathrm{d}s + \Gamma_a, \end{aligned} \tag{56}$$

where

$$\Gamma_a = \int_{t_n}^{t} \left(\left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(t) \cdot \dot{\boldsymbol{\epsilon}}(s) - \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(s) \cdot \dot{\boldsymbol{\epsilon}}(t) \right) \mathrm{d}s. \tag{57}$$

Furthermore, by the application of the Leibniz's rule it follows from (56) that

$$\Delta\boldsymbol{\epsilon} \cdot \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{t_n}^{t} \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(s) \cdot \Delta\boldsymbol{\epsilon}\,\mathrm{d}s + \frac{1}{2}\Gamma_a. \tag{58}$$

By definitions (55), the second order dissipation writes

$$\Delta_2\Gamma = \frac{1}{2}\int_{t_n}^{t_{n+1}} \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \dot{\boldsymbol{\epsilon}}(t) \cdot \Delta\boldsymbol{\epsilon}\,\mathrm{d}t + \frac{1}{2}\Gamma_a. \tag{59}$$

Let us consider now within the set $\mathscr{H}$ the radial strain path

$$\boldsymbol{\epsilon}^{\mathrm{r}}(t) = \boldsymbol{\epsilon}_n + \frac{t - t_n}{t_{n+1} - t_n}(\boldsymbol{\epsilon}_{n+1} - \boldsymbol{\epsilon}_n). \tag{60}$$

The path-dependent term evaluated to the second-order along a radial path departing from $\boldsymbol{\epsilon}_n$, after straightforward integration is specified as the sum of

$$\Delta_1^{\mathrm{r}}\Gamma = \boldsymbol{\tau}_n \cdot \Delta\boldsymbol{\epsilon}, \qquad \Delta_2^{\mathrm{r}}\Gamma = \frac{1}{2}\Delta\boldsymbol{\epsilon} \cdot \left.\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\right|_n \Delta\boldsymbol{\epsilon}. \tag{61}$$

It can be proved that:

> If the path-dependent function $\Delta\Gamma$ satisfies the symmetry condition
> $$\Gamma_a = 0, \tag{62}$$
> where Γ_a is defined by Eq. (57), and the argument of $\Delta\Gamma$ is convex in $\dot{\epsilon}$, then $\Delta\Gamma$ evaluated to the second order is minimized on a radial path
> $$\Delta\Gamma \geqslant \Delta_1^{\mathrm{r}}\Gamma + \Delta_2^{\mathrm{r}}\Gamma + \mathrm{o}(\|\Delta t\|^2). \tag{63}$$

This can be shown by observing that the path dependent term becomes

$$\Delta\Gamma = \int_{t_n}^{t_{n+1}} \boldsymbol{\tau}_n \cdot \dot{\boldsymbol{\epsilon}}\,\mathrm{d}t + \frac{1}{2}\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\bigg|_n \dot{\boldsymbol{\epsilon}}(t)\cdot\Delta\boldsymbol{\epsilon}(t)\,\mathrm{d}t + \mathrm{o}(\|\Delta t\|^2) = \int_{t_n}^{t_{n+1}} \boldsymbol{\tau}\left(\boldsymbol{\epsilon}_n + \frac{1}{2}\Delta\boldsymbol{\epsilon}\right)\cdot\dot{\boldsymbol{\epsilon}}(t)\,\mathrm{d}t + \mathrm{o}(\|\Delta t\|^2). \tag{64}$$

Replacing $\boldsymbol{\epsilon}_n$ with $\boldsymbol{\epsilon}_n + \frac{1}{2}\Delta\boldsymbol{\epsilon}$, and by the convexity property of the argument of $\Delta\Gamma$, it follows that

$$\Delta\Gamma \geqslant \boldsymbol{\tau}\left(\boldsymbol{\epsilon}_n + \frac{1}{2}\Delta\boldsymbol{\epsilon}\right)\cdot\Delta\boldsymbol{\epsilon} + \mathrm{o}(\|\Delta t\|^2) = \boldsymbol{\tau}_n\cdot\Delta\boldsymbol{\epsilon} + \frac{1}{2}\frac{\mathrm{d}\boldsymbol{\tau}}{\mathrm{d}\boldsymbol{\epsilon}}\bigg|_n \Delta\boldsymbol{\epsilon}\cdot\Delta\boldsymbol{\epsilon} + \mathrm{o}(\|\Delta t\|^2), \tag{65}$$

which in view of Eq. (61) proves Eq. (63).

Let us define the change of strain energy along a radial path

$$\Delta^{\mathrm{r}}\Psi = \Delta\Phi + \Delta_1^{\mathrm{r}}\Gamma + \Delta_2^{\mathrm{r}}\Gamma + \mathrm{o}(\|\Delta t\|^2). \tag{66}$$

If Eq. (63) holds, in view of the fact that $\Delta\Phi$ is path-independent, the increment of mechanical work $\Delta\Psi$ in $\mathscr{H}$ is never less than the the increment of mechanical work evaluated on a radial strain path

$$\boxed{\Delta\Psi \geqslant \Delta\Psi^{\mathrm{r}}.} \tag{67}$$

Eq. (67) holds if the argument of $\Delta\Gamma$ is convex in $\dot{\boldsymbol{\epsilon}}$ for every $\boldsymbol{\epsilon}$, and under hypothesis (62). Therefore, it has been shown that among all possible strain paths emanating from $\boldsymbol{\epsilon}_n$ to $\boldsymbol{\epsilon}_{n+1}$ and belonging to $\mathscr{H}$, the radial strain path $\boldsymbol{\epsilon}^{\mathrm{r}}$ realizes the minimum change of the mechanical work $\Delta\Psi$ in the set $\mathscr{H}$. However, although radial strain paths have been shown to be less energy-consuming than nonradical paths, they can still be unstable. Thus stability theorems such as the ones applied in Nguyen (1993) and Petryk (2002) cannot be applied to the present class of materials.

Moreover, it is possible to establish a relationship between the symmetry condition (62) and the coaxiality condition (49) for the existence of a strain potential. To this purpose, let us observe that the rate of the stress τ can be written as

$$\dot{\tau} = \mathbb{Z}\dot{\boldsymbol{\epsilon}}, \tag{68}$$

where $\mathbb{Z}$ denotes the tangent tensor associated to the stress τ

$$\mathbb{Z} = \frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}} \otimes \mathbb{E}\boldsymbol{\epsilon} + Y\frac{\mathrm{d}^2\omega}{\mathrm{d}\boldsymbol{\epsilon}^2}. \tag{69}$$

The symmetry condition (62) can be rewritten as

$$\int_{t_n}^{t} \delta(\tau(t);\dot{\boldsymbol{\epsilon}}(s))\,\mathrm{d}s = \int_{t_n}^{t} \delta(\tau(s);\dot{\boldsymbol{\epsilon}}(t))\,\mathrm{d}s, \tag{70}$$

which is equivalent to a weak condition for the potential of the stress τ to exist along the step $[t_n, t]$. In particular, if the fourth order tensor $\mathbb{Z}$ is symmetric, then Eq. (70) holds; suppose in addition that the matrix $\frac{d^2\omega}{d\epsilon^2}$ is symmetric, then the tensor $\frac{d\omega}{d\epsilon} \otimes \mathbb{E}\epsilon$ is symmetric. The converse is not generally true, i.e. the only symmetry of the tangent tensor $\mathbb{H}$ (47) does not imply the symmetry of the tangent tensor $\mathbb{Z}$ (69). In conclusion, the coaxiality condition (49) is equivalent to the symmetry condition of the tensor $\mathbb{Z}$ (69) provided that the fourth order tensor $\frac{d^2\omega}{d\epsilon^2}$ is symmetric. For instance, this circumstance occurs in case of associative damage governed by the coaxiality property (49), because $\frac{d^2\omega}{d\epsilon^2}$ results to be proportional to $\mathbb{E}$, the elasticity tensor.

6. Nonconvex potentials in a one-dimensional case

A monotonic tensile loading history is prescribed for a one-dimensional bar of infinite length. Let the bar be subject to a positive homogeneous strain state ϵ. The loading history is holonomic by hypothesis, that is the damage is assumed to evolve monotonically during the loading. Hence the stepwise holonomic approximation is not invoked. The equivalent strain definition (4) is assumed, which specializes here to $\bar{\epsilon} = \epsilon$. Because all the quantities involved in Eq. (47) are scalars, the symmetry condition (47) is satisfied. The damage rule (12) (Peerlings et al., 1996; Geers, 1997)

$$\omega(\epsilon) = \begin{cases} 0 & \text{if } \kappa_i \leqslant \epsilon, \\ 1 - \left(\frac{\kappa_i}{\epsilon}\right)^{\beta}\left(\frac{\kappa_f - \epsilon}{\kappa_f - \kappa_i}\right)^{\alpha} & \text{if } \kappa_i < \epsilon \leqslant \kappa_f, \\ 1 & \text{if } \epsilon > \kappa_f \end{cases} \tag{71}$$

is adopted. According to Eq. (71), the first derivative of the damage becomes

$$\frac{d\omega}{d\epsilon} = \begin{cases} 0 & \text{if } \epsilon < \kappa_i, \\ (1 - \omega(\epsilon))\left(\frac{\beta}{\epsilon} + \frac{\alpha}{\kappa_f - \epsilon}\right) & \text{if } \kappa_i \leqslant \epsilon \leqslant \kappa_f, \\ 0 & \text{if } \epsilon > \kappa_f. \end{cases} \tag{72}$$

Replacing Eqs. (71) and (72) in Eq. (31) the path dependent term with $\alpha = \beta = 1$ writes

$$\int_0^{\epsilon} \frac{d\omega}{d\tilde{\epsilon}} \frac{1}{2} E\tilde{\epsilon}^2 \, d\tilde{\epsilon} = \begin{cases} 0 & \text{if } \epsilon < \kappa_i, \\ \frac{1}{2}\frac{E\kappa_i\kappa_f}{\kappa_f - \kappa_i}(\epsilon - \kappa_i) & \text{if } \kappa_i \leqslant \epsilon \leqslant \kappa_f, \\ \frac{1}{2}\frac{E\kappa_i\kappa_f}{\kappa_f - \kappa_i}(\kappa_f - \kappa_i) & \text{if } \epsilon > \kappa_f. \end{cases} \tag{73}$$

Finally, the explicit analytical expression of the strain potential (31) becomes

$$\Psi(\epsilon) = \begin{cases} \frac{1}{2}E\epsilon^2 & \text{if } \epsilon < \kappa_i, \\ (1 - \omega(\epsilon))\frac{1}{2}E\epsilon^2 + \frac{1}{2}\frac{E\kappa_i\kappa_f}{\kappa_f - \kappa_i}(\epsilon - \kappa_i) & \text{if } \kappa_i \leqslant \epsilon \leqslant \kappa_f, \\ (1 - \omega(\epsilon))\frac{1}{2}E\epsilon^2 + \frac{1}{2}\frac{E\kappa_i\kappa_f}{\kappa_f - \kappa_i}(\kappa_f - \kappa_i) & \text{if } \epsilon > \kappa_f. \end{cases} \tag{74}$$

The stress conjugated to the strain potential (31) is the expected one

$$\sigma = \frac{d\Psi}{d\epsilon} = (1 - \omega(\epsilon))E\epsilon \tag{75}$$

and can be visualized in Fig. 10 (continuous line), where the Young's modulus $E = 20{,}000$ N/mm^2, and the initial and final thresholds are $\kappa_i = 0.0002$ and $\kappa_f = 0.0125$.

As shown in Fig. 11A and B, in the present one-dimensional example, the strain energy Ψ (74) is convex in ϵ in the elastic range, when $\epsilon < \kappa_i$, and concave in ϵ as soon as the damaging process begins, at $\epsilon = \kappa_i$. Note however that the slope of Ψ at incipient damage is continuous, see Fig. 11B (continuous line). The strain potential (74) is similar to the strain super-potential considered by Mistakidis and Panagiotopoulos

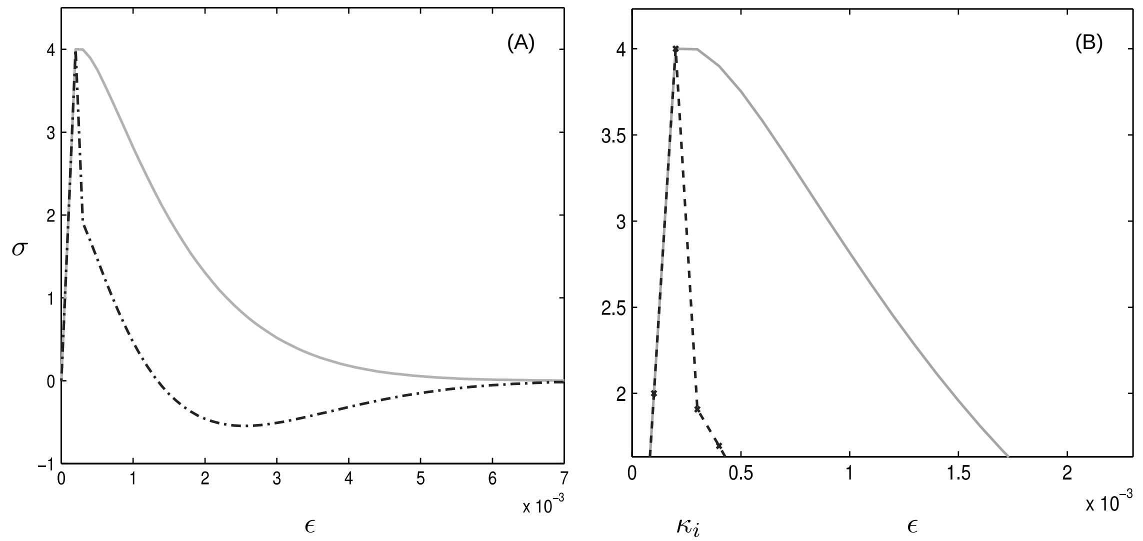

Fig. 10. (A) Strain energies Ψ Eq. (74) (continuous line) and Ψ_1 Eq. (76) (dashed line) versus ϵ in the one-dimensional tensile bar. (B) Details of Ψ and Ψ_1 versus ϵ at incipient damage in the one-dimensional tensile bar.

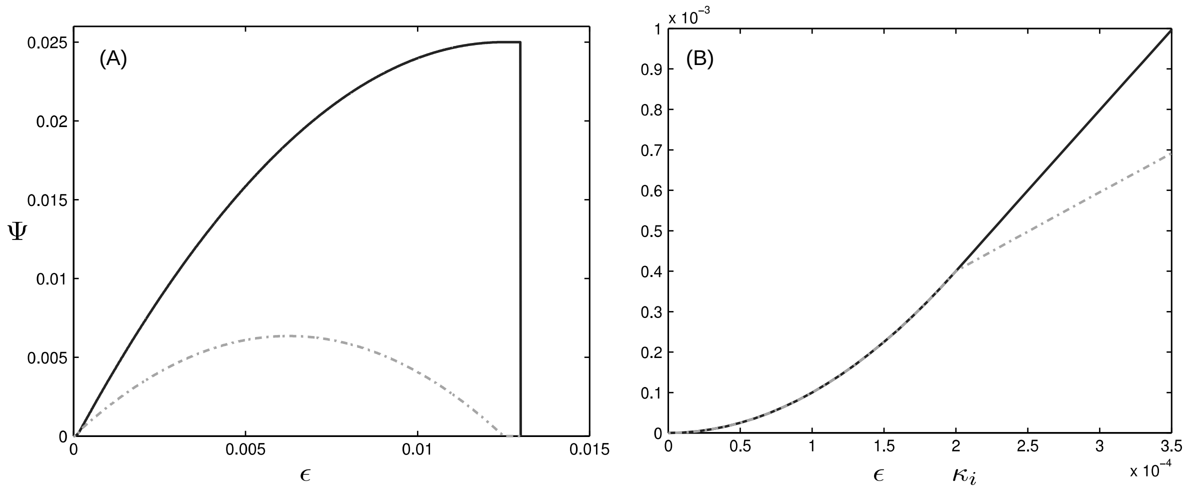

Fig. 11. (A) Stresses $\sigma = \mathrm{d}\Psi/\mathrm{d}\epsilon$ (continuous line) and $\sigma_1 = \mathrm{d}\Psi_1/\mathrm{d}\epsilon$ (dashed line) versus the strain ϵ in the one-dimensional tensile bar (B) Detail of σ and σ_1 versus ϵ at incipient damage; if the integral term (73) is included or neglected, σ and σ_1 are continuous and discontinuous at $\kappa = \kappa_\mathrm{i}$.

(1998) for a nonmonotonic stress–strain law and to the free energy potential proposed for a microscopic cohesive stress–displacement law in Nguyen and Ortiz (2002). If convexity is lacking, equilibrium states are not necessarily stable and the knowledge of the second order variation of the strain potential would be required to go farther and capture (local) minimizers. In several contexts, nonconvex strain energies have been recognized to reflect the presence of multiple phases or microstructures, for instance Miehe et al. (2002) and Ortiz and Repetto (1999). The present results highlight the common basis underlying damaging and multiphase materials.

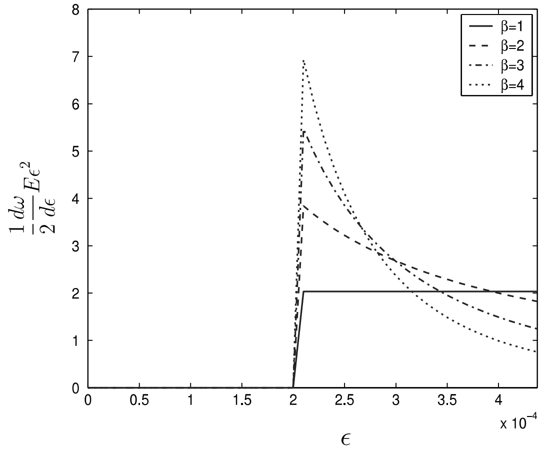

Fig. 12. The term $1/2\,\mathrm{d}\omega/\mathrm{d}\epsilon E\epsilon^2$ versus ϵ in the one-dimensional tensile bar with $\alpha = 1$, and β increasing; the discontinuity at $\epsilon = k_{\mathrm{i}}$ reflects the discontinuity of $\mathrm{d}\omega/\mathrm{d}\epsilon$.

To highlight the practical influence of the argument of the integral at the l.h.s. of Eq. (73), the latter term has been plotted in Fig. 12 for fixed $\alpha = 1$ and β increasing from 1 to 4. A discontinuity occurs at $\epsilon = \kappa_{\mathrm{i}}$ which becomes sharper as β increases. If the path-dependent term (73) is neglected, the strain function

$$\Psi_1(\epsilon) = (1 - \omega(\epsilon))\frac{1}{2}E\epsilon^2 \tag{76}$$

is obtained, see Fig. 11. A detail of the strain function Ψ_1 at incipient damage reveals that its slope changes at incipient damage, Fig. 11B. The stress conjugated to Ψ_1

$$\sigma_1 = \frac{\mathrm{d}\Psi_1}{\mathrm{d}\epsilon} = (1 - \omega(\epsilon))E\epsilon - \frac{\mathrm{d}\omega}{\mathrm{d}\epsilon}\frac{1}{2}E\epsilon^2 \tag{77}$$

has been plotted in Fig. 10A (continuous line). As shown in Fig. 10B, σ_1 exhibits a discontinuity at incipient damage due to the discontinuity of the first derivative of the damage (Fig. 9). Consequently, the presence of $\Delta\Gamma$ is necessary in order to reproduce physically consistent stress–strain relationships. Note that in the GSM context, the free energy functional is usually assumed to be convex in both damage and strain. Therefore, once one of the variables is fixed, the minimization with respect to the other variable can be performed (e.g. Florez-Lopez et al., 1994). By the integration of higher order stress–strain laws, in Chang et al. (2002), strain energies have been obtained where, besides the standard terms, also derivatives of the damage with respect to the strain appear, but without an explicit analysis of the numerical influence of these damage derivatives on the stress–strain law.

7. Nonlocal materials

In the context of nonlocal models for elastodamaging materials, many authors have considered the damage parameter to be governed by nonlocal definitions of the equivalent strain (Pijaudier-Cabot, 1995; de Borst et al., 1997; Jirásek and Patzák, 2002). For instance, a nonlocal integral equivalent strain field $\{\bar{\epsilon}\}$ can be obtained by an integral average of the local field $\bar{\epsilon}$

$$\{\bar{\epsilon}\}(\mathbf{x}) = \frac{1}{V_r(\mathbf{x})}\int_V w(\mathbf{x},\mathbf{y})\bar{\epsilon}(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{78}$$

In Eq. (78), $w(\mathbf{x},\mathbf{y})$ is a weighting function, for instance the Gauss function $e^{-k^2\|\mathbf{x}-\mathbf{y}\|^2/\ell^2}$, where k is a parameter and ℓ indicates a characteristic length over which w vanishes. The symbol

$$V_r(\mathbf{x}) = \int_V w(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y} \tag{79}$$

is the reference volume. Besides integral models, also several successful formulations based on implicit gradient techniques have been developed in Peerlings et al. (1996) and Geers (1997) by expanding in Taylor series the integrand of (78) under the hypothesis that the integration domain is symmetric. The nonlocal variable $\{\bar{\epsilon}\}$ is so approximated through the following implicit gradient formula:

$$\{\bar{\epsilon}\}(\mathbf{x}) \approx \bar{\epsilon}(\mathbf{x}) + c^2\nabla^2\{\bar{\epsilon}\}(\mathbf{x}), \tag{80}$$

together with the boundary conditions

$$\nabla\{\bar{\epsilon}\}(\mathbf{x})\cdot\mathbf{n} = 0, \tag{81}$$

on the boundary ∂V, where ∇^2 is the spatial Laplacian operator and c is a diffusive length that is related to the characteristic length ℓ. In particular, these Authors have proposed multifield formulations where the nonlocal definition of the equivalent strain is assumed as an independent variable and the implicit gradient definition (80) is solved together with the equilibrium equation. These multifield formulations based on implicit gradient formulations are particularly appreciated since they have been revealed to be more computationally stable than other gradient techniques (Geers, 1997). For instance, approximating $\{\bar{\epsilon}\}$ by Eq. (80) as an independent field reduces the occurrence of oscillations of the computed stress profiles with respect to other (explicit) gradient formulations.

Because both the integral and the derivative operators are linear operators, all the results established in the previous sections can be extended to the case of nonlocal definitions of the equivalent strain. To this purpose, let the damage and its threshold be nondecreasing functions of the nonlocal equivalent strain, i.e., respectively,

$$\omega_{n+1} = \max\{\omega_n, \omega(\{\bar{\epsilon}\}_{n+1})\}, \qquad \kappa_{n+1} = \max\{\kappa_n, \{\bar{\epsilon}\}_{n+1}\}, \tag{82}$$

where $\{\bar{\epsilon}\}(\boldsymbol{\epsilon}_{n+1}) \equiv \{\bar{\epsilon}\}_{n+1}$. The loading function becomes

$$g_{n+1} = \{\bar{\epsilon}\}_{n+1} - \kappa_{n+1}, \tag{83}$$

where relationships (82) have been assumed.

7.1. An enhanced general variational framework

The results obtained in the previous sections are here embedded into variational formulations for elastodamaging materials of the nonlocal type. A body is assumed obeying an elastodamaging stress–strain law, where the damage and its threshold are governed by the nonlocal equivalent strain according to Eqs. (82) and (83). The strain energy function (23) writes

$$\Psi_{n+1} = (1-\omega_{n+1})Y_{n+1} - (1-\omega_n)Y_n + \int_{\boldsymbol{\epsilon}_n}^{\boldsymbol{\epsilon}_{n+1}} \frac{\partial\omega}{\partial\{\bar{\epsilon}\}}\frac{\partial\{\bar{\epsilon}\}}{\partial\boldsymbol{\epsilon}}Y\,\mathrm{d}\boldsymbol{\epsilon} + \Psi_n. \tag{84}$$

Let us consider the nonlocal relationships of integral and implicit gradient type (78) and (80) and define a nonlocal functional $\mathscr{R}$ of the strain

$$\mathscr{R}(\boldsymbol{\epsilon}_{n+1})(\mathbf{x}) = \begin{cases} \frac{1}{V_r(\mathbf{x})}\int_V w(\mathbf{x},\mathbf{y})\bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}(\mathbf{x}))\,\mathrm{d}\mathbf{y}, \\ \{\bar{\epsilon}\}(\boldsymbol{\epsilon}_{n+1}(\mathbf{x})) = \bar{\epsilon}(\boldsymbol{\epsilon}_{n+1}(\mathbf{x})) + c^2\nabla^2\{\bar{\epsilon}\}(\boldsymbol{\epsilon}_{n+1}(\mathbf{x})) \quad \text{in } V, \end{cases} \tag{85}$$

where the first alternative refers to an integral nonlocal relationship whereas the second one describes an implicit gradient formulation. In definition (85), the strain function $\bar{\epsilon}_{n+1}$ is any of the equivalent strain definitions reported in Section 2, and the field $\{\bar{\epsilon}\}_{n+1}$ is assumed to be an additional independent variable. Let us omit below the **x**-dependence of the nonlocal operators. The equality

$$\{\bar{\epsilon}\}_{n+1} = \mathscr{R}(\boldsymbol{\epsilon}_{n+1}) \tag{86}$$

represents a constraint between the nonlocal field $\{\bar{\epsilon}\}_{n+1}$ and the strain field $\bar{\epsilon}_{n+1}$. It is then possible to introduce a multifield functional containing the strain functional (39) where the constraint (86) is weakly enforced by a Lagrangian multiplier field λ_{n+1}. So the following Lagrangian functional depending on $\mathbf{X}_{n+1} \equiv (\mathbf{u}_{n+1}, \boldsymbol{\epsilon}_{n+1}, \boldsymbol{\sigma}_{n+1}, \{\bar{\epsilon}\}_{n+1}, \lambda_{n+1})$

$$\mathscr{L}(\mathbf{X}_{n+1}) = \mathscr{F}(\mathbf{X}_{n+1}) + \int_V \lambda_{n+1}(\{\bar{\epsilon}\}_{n+1} - \mathscr{R}(\boldsymbol{\epsilon}_{n+1}))\,\mathrm{d}V \tag{87}$$

can be considered. In Eq. (87),

$$\mathscr{F}(\mathbf{X}_{n+1}) = \int_V \Phi_{n+1} - \Phi_n + \int_{\boldsymbol{\epsilon}_n}^{\boldsymbol{\epsilon}_{n+1}} \frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}} Y\,\mathrm{d}\boldsymbol{\epsilon} + \Psi_n - \mathbf{b}_{n+1}\cdot\mathbf{u}_{n+1} + \boldsymbol{\sigma}_{n+1}\cdot(\nabla^s\mathbf{u}_{n+1} - \boldsymbol{\epsilon}_{n+1})\,\mathrm{d}V \tag{88}$$

is the Hu–Washizu functional (23) written in the case of nonlocal definition of the equivalent strain, and, by the chain rule, $\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}} \equiv \frac{\partial\omega}{\partial\{\bar{\epsilon}\}}\frac{\partial\{\bar{\epsilon}\}}{\partial\boldsymbol{\epsilon}}$. Equating to zero the first variations of function (87) leads to the weak forms of the stationarity equations

$$\delta\mathscr{L}(\mathbf{X}_{n+1};\hat{\mathbf{u}}) = \int_V \boldsymbol{\sigma}_{n+1}\cdot\nabla^s\hat{\mathbf{u}}_{n+1} - \mathbf{b}_{n+1}\cdot\hat{\mathbf{u}}\,\mathrm{d}V = 0, \tag{89}$$

$$\delta\mathscr{L}(\mathbf{X}_{n+1};\hat{\boldsymbol{\epsilon}}) = \int_V \left((1-\omega_{n+1})\mathbb{E}\boldsymbol{\epsilon}_{n+1} + \left.\frac{\mathrm{d}\omega}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1} Y_{n+1} - \boldsymbol{\sigma}_{n+1} - \lambda_{n+1}\left.\frac{\mathrm{d}\mathscr{R}(\boldsymbol{\epsilon})}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1}\right)\cdot\hat{\boldsymbol{\epsilon}}\,\mathrm{d}V = 0, \tag{90}$$

$$\delta\mathscr{L}(\mathbf{X}_{n+1};\hat{\boldsymbol{\sigma}}) = \int_V (\nabla^s\mathbf{u}_{n+1} - \boldsymbol{\epsilon}_{n+1})\cdot\hat{\boldsymbol{\sigma}}\,\mathrm{d}V = 0, \tag{91}$$

$$\delta\mathscr{L}(\mathbf{X}_{n+1};\{\hat{\epsilon}\}) = \int_V \left(-\left.\frac{\partial\omega}{\partial\{\bar{\epsilon}\}}\right|_{n+1} Y_{n+1} + \lambda_{n+1}\right)\hat{\bar{\epsilon}}\,\mathrm{d}V = 0, \tag{92}$$

$$\delta\mathscr{L}(\mathbf{X}_{n+1};\hat{\lambda}) = \int_V (\{\bar{\epsilon}\}_{n+1} - \mathscr{R}(\boldsymbol{\epsilon}_{n+1}))\hat{\lambda}\,\mathrm{d}V = 0, \tag{93}$$

for any admissible set of variations $(\hat{\mathbf{u}}, \hat{\boldsymbol{\epsilon}}, \hat{\boldsymbol{\sigma}}, \hat{\bar{\epsilon}}, \hat{\lambda})$. The strong forms of the stationarity conditions of $\mathscr{L}$ realize Problem P3 below:

P3. Find $\mathbf{X}_{n+1}$ such that on V:

$$\operatorname{div}\boldsymbol{\sigma}_{n+1} + \mathbf{b}_{n+1} = \mathbf{0}, \tag{94a}$$

$$\boldsymbol{\sigma}_{n+1} = (1-\omega_{n+1})\mathbb{E}\boldsymbol{\epsilon}_{n+1} + \left.\frac{\partial\omega}{\partial\{\bar{\epsilon}\}}\right|_{n+1} \left.\frac{\partial\{\bar{\epsilon}\}}{\partial\boldsymbol{\epsilon}}\right|_{n+1} Y_{n+1} - \lambda_{n+1}\left.\frac{\mathrm{d}\mathscr{R}(\boldsymbol{\epsilon})}{\mathrm{d}\boldsymbol{\epsilon}}\right|_{n+1}, \tag{94b}$$

$$\nabla^{s}\mathbf{u}_{n+1} = \boldsymbol{\epsilon}_{n+1}, \tag{94c}$$

$$\lambda_{n+1} = \left.\frac{\partial\omega}{\partial\{\bar{\epsilon}\}}\right|_{n+1} Y_{n+1}, \tag{94d}$$

$$\{\bar{\epsilon}\}_{n+1} = \mathscr{R}(\boldsymbol{\epsilon}_{n+1}), \tag{94e}$$

where $\omega_{n+1} = \max\{\omega_n, \omega(\{\bar{\epsilon}\}_{n+1})\}$.

By replacing Eqs. (94d) and (94e) in Eq. (94b), the stress $\boldsymbol{\sigma}_{n+1}$ Eq. (22) is recovered. Thus, the system of Eq. (94) condenses into the concise form:

P4. Find $\mathbf{X}_{n+1}$ such that on V

$$\operatorname{div}\boldsymbol{\sigma}_{n+1} + \mathbf{b}_{n+1} = \mathbf{0}, \tag{95a}$$

$$\{\bar{\epsilon}\}_{n+1} - \mathscr{R}(\boldsymbol{\epsilon}_{n+1}) = 0, \tag{95b}$$

where

- $\boldsymbol{\sigma}_{n+1} = (1-\omega_{n+1})\mathbb{E}\boldsymbol{\epsilon}_{n+1}$,
- $\nabla^{s}\mathbf{u}_{n+1} = \boldsymbol{\epsilon}$,
- $\omega_{n+1} = \max\{\omega_n, \omega(\{\bar{\epsilon}\}_{n+1})\}$.

The above Lagrangian formulation can be discretized according to the FE method and leads to a symmetric solving system (Benvenuti et al., submitted for publication). It can be also shown that the boundary condition (81), typically assumed in the implicit gradient formulations, can be obtained via variational arguments from the stationarity conditions of the Lagrangian functional (87). The same set of equations of Problem P4 has been proposed in a broad number of multifield formulations of implicit gradient type by Peerlings et al. (1996) and Geers (1997). Consequently, the Lagrangian approach presented in this section can provide a variational basis for these existing formulations.

8. Conclusions

At a material point, a relationship between total damage and total strain is deduced by considering a stepwise holonomic damage evolution. By integrating the mechanical work in the strain space, the following results have been reached.

- Incremental strain energies have been determined for elastodamaging materials; in particular, a path-independent strain energy has been obtained under the hypothesis of associative damage evolution (Sections 3 and 4).
- A coaxiality condition for the incremental strain energy to be potential has been derived, and its connections with the associativity of the damage evolution have been discussed (Section 4).
- Among all admissible strain paths, the radial one has been shown to minimize the increment of the mechanical work in the strain space, provided that a suitable additional symmetry condition, anyway related to the coaxiality condition, is fulfilled (Section 5).
- For a one-dimensional bar under a strictly monotonic loading, the present elastodamaging materials may be regarded as nonlinear elastic materials with nonconvex strain energy, in analogy with microstructured and geometrically nonlinear materials (Section 6).
- A new multifield Lagrangian formulation has been presented providing a unified variational framework for nonlocal materials (Section 7).

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